Int. J. Solids Structures, 1968, Vol. 4, pp. 1233 to 1242. Pergamon Press. Printed in Great Britain

# STABILITY IN LINEAR ELASTICITY*

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**Abstract**—Conditions are established ensuring the continuous dependence on the initial data of the equilibrium solution and certain other classes of solution to the elastodynamic initial boundary value problem. The method of proof depends upon logarithmic convexity arguments and is notable for the absence of any definiteness condition on the elasticities. In some cases the conclusions reached differ from corresponding ones derived according to classical Lyapounov stability theory and examples are given to illustrate this difference.

## 1. INTRODUCTION

THIS paper examines the stability of equilibrium solutions to problems in linear elasticity according to an extended Lyapounov definition. Instead, however, of investigating stability by the customary procedure known as Lyapounov's second method, we adopt an entirely different technique based on convexity arguments. These are currently of wide use in the allied field of non-well posed problems (see, e.g., Payne [9] and the references there cited) and in the present context lead to conditions stabilizing an otherwise unstable solution. Put differently, they lead to conditions ensuring that the problem becomes well posed. Thus, we are able to prove that equilibrium solutions are always stable provided the perturbations lie in a suitable class of uniformly bounded functions and provided also that the elasticities satisfy a particular symmetry. No other restriction need be imposed on the elasticities. This fact is all the more remarkable since without such boundedness conditions a non-positive-definite strain energy alone is sufficient to produce asymptotic instability. Of course, when the strain energy is positive-definite our conclusions merely repeat some of those already known in the literature.

It should be remarked that although we are concerned only with the question of stability of the equilibrium solution under dynamical perturbations, in this paper we actually establish continuous dependence on the initial data for solutions of certain classes of initial-boundary value problems of dynamical elasticity. This means that our results apply equally well for perturbations from any given dynamical state and not merely for perturbations about equilibrium.

Unless stated otherwise stability in this paper is defined in a slightly more general sense than is usual in the majority of treatments. We take the period of investigation to be a *half-open* finite time interval and not a *closed* interval as in the classical definition. We shall see that many solutions which might be classified as unstable in the usual sense will be considered stable according to the definition used in this paper.

* This research was supported in part by the National Science Foundation under Grant GP-5882.

Our conclusions appear at first glance to contradict a recent theorem of Gilbert and Knops [1] asserting the equivalence of stability and boundedness (when these terms are properly defined). However, this disagreement is easily resolved.

Throughout, we refer all equations to cartesian coordinates $x_i$, and we adopt the convention of summing over repeated suffixes whose values are 1, 2 and 3. A classical solution is always assumed to exist although the following results could be extended in a straightforward manner to include certain generalized solutions.

## 2. BASIC EQUATIONS AND STABILITY CONCEPTS

We consider a linear elastic solid occupying a closed regular three-dimensional region of euclidean space $B$ bounded by a smooth surface $\partial B$. Because of linearity we need consider only the stability of the null solution of the homogeneous equations

$$\frac{\partial}{\partial x_j}\left\{c_{ijkl}\frac{\partial u_k}{\partial x_l}\right\} = \rho\frac{\partial^2 u_i}{\partial t^2} \quad \text{in} \quad B\times(0, T], \tag{2.1}$$

$$c_{ijkl} = c_{klij}, \tag{2.2}$$

subject to the boundary conditions

$$\begin{aligned} u_i &= 0 \quad \text{on} \quad \overline{\partial B_1}\times[0, T], \\ c_{ijkl}\frac{\partial u_k}{\partial x_l}n_j &= 0 \quad \text{on} \quad \partial B_2\times[0, T] \end{aligned} \tag{2.3}$$

where $\partial B_1$ and $\partial B_2$ are disjoint subsets of $\partial B$ such that $\partial B = \overline{\partial B_1} \cup \partial B_2$. The closure of $\partial B_1$ is denoted by $\overline{\partial B_1}$, the closed time interval of length $T$ is $[0, T]$, and the cartesian product of the sets $B$ and $[0, T]$ is denoted by $B\times[0, T]$. In addition, $u_i$ denotes the cartesian components of displacement, $t$ is the time variable, $n_i$ are the cartesian components of the outward normal on $\partial B_2$, $\rho$ is the non-homogeneous density, assumed positive, and $c_{ijkl}$ are the non-homogeneous elasticities. To avoid a clumsy notation we display here for the only time the dependence of the above quantities upon their arguments:

$$u_i = u_i(\mathbf{x}, t), \qquad \rho = \rho(\mathbf{x}), \qquad c_{ijkl} = c_{ijkl}(\mathbf{x}). \tag{2.4}$$

Observe that because (2.2) is the only symmetry imposed on the elasticities the subsequent calculations remain valid in the theory of small elastic deformations superposed upon large. Also, even though (2.3) are the only boundary conditions discussed other standard types could be equally included.

The prescription of initial conditions more appropriately forms part of our

*Definition of stability*. The null solution is stable under perturbations $u_i$ satisfying (2.1)–(2.3) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\left[\int_{B(0)} \rho u_i u_i \,\mathrm{d}v + Q\right] < \delta \tag{2.5}$$

implies

$$\sup_{0\le t< T}\left[\int_{B(t)} \rho u_i u_i \,\mathrm{d}v + Q\right] < \varepsilon. \tag{2.6}$$

Here, $B(t)$ denotes integration over the volume of the body at time $t$, while $Q$ is an appropriately chosen positive functional of the initial data which tends to zero as the initial data tends to zero. Its precise form will be specified later. We say that a solution is *unstable* when it is not stable.

In this definition, the period of investigation is the half-open time interval of finite length $T$. If instead, the closed interval of length $T$ had been taken in (2.6), then the definition would have reduced to a version of Movchan's [7] generalization of the classical definition due to Lyapounov. We shall see by means of an example that our conclusions are highly sensitive to the choice in (2.6) of the half-open or closed time interval.

## 3. STABILITY ANALYSIS

We now employ convexity arguments to prove:

THEOREM 3.1. *For the system of equations* (2.1)–(2.3) *the null solution is stable provided the perturbations satisfy the uniform boundedness condition*

$$\int_{B(t)} \rho u_i u_i \, \mathrm{d}v < M^2 \tag{3.1}$$

*for some positive bounded constant M.*

*Proof.* Consider the function $G(t)$ defined by

$$G(t) = \log[F(t)+Q]+t^2, \tag{3.2}$$

where

$$F(t) = \int_{B(t)} \rho u_i u_i \, \mathrm{d}v. \tag{3.3}$$

We shall establish the convexity of $G(t)$ on $[0, T]$; that is, we shall prove that

$$(F+Q)^2\frac{\mathrm{d}^2G}{\mathrm{d}t^2} \equiv (F+Q)\frac{\mathrm{d}^2F}{\mathrm{d}t^2}-\left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)^2+2(F+Q)^2 \geq 0,\ 0 \leq t \leq T. \tag{3.4}$$

Now,

$$\frac{\mathrm{d}F}{\mathrm{d}t} = 2\int_{B(t)} \rho u_i \frac{\partial u_i}{\partial t}\mathrm{d}v,$$

and

$$\frac{\mathrm{d}^2F}{\mathrm{d}t^2} = 2\int_{B(t)} \left(\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}+\rho u_i\frac{\partial^2 u_i}{\partial t^2}\right)\mathrm{d}v$$

which, with the help of (2.1)–(2.3), may be written

$$\frac{\mathrm{d}^2F}{\mathrm{d}t^2} = 2\int_{B(t)} \left(\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}-c_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}\right)\mathrm{d}v. \tag{3.5}$$

Since the energy $E(t)$, defined as

$$E(t) = \frac{1}{2}\int\limits_{B(t)} \left(\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} + c_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}\right) \mathrm{d}v, \tag{3.6}$$

is time-independent (i.e. $E(0) = E(t)$), we may use Schwarz's inequality to obtain

$$(F+Q)\frac{\mathrm{d}^2F}{\mathrm{d}t^2} - \left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)^2 \geq -4E(0)[F+Q] + 4Q\int\limits_{B(t)} \rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}\mathrm{d}v \geq -2(F+Q)^2 \tag{3.7}$$

provided $Q$ is chosen to satisfy $Q \geq 2E(0)$. Thus (3.4) is established. Let us choose, in particular,

$$Q = 2\max[E(0), 0] \tag{3.8}$$

and note that for a positive initial energy tending to zero then $Q \to 0$. From the convexity of $G(t)$ it immediately follows that

$$G(t) \leq (t/T)G(T) + (1-t/T)G(0), \qquad 0 \leq t \leq T$$

i.e.

$$F(t) + Q \leq \mathrm{e}^{t(T-t)}[F(T)+Q]^{t/T}[F(0)+Q]^{1-t/T}, \qquad 0 \leq t \leq T. \tag{3.9}$$

Since all terms on the right of (3.9) remain bounded it follows that for $0 \leq t < T$ arbitrarily small values of $F(0)+Q$ imply arbitrarily small values of $F(t)+Q$; the theorem is thus proved.

Notice that (3.9) fails to predict either the stability or instability of the null solution in the closed interval in the sense that $F(0)+Q \to 0$ does not imply $\sup_{0\leq t\leq T} F(t)+Q \to 0$. In fact at $t = T$(3.9) reduces to an identity.

A special case of Theorem 3.1 has been found previously by Zorski [13], Slobokin [12] and Knops and Wilkes [5]* for the displacement boundary value problem provided the equilibrium equations corresponding to (2.1) are strongly elliptic, or alternatively, that the strain energy is positive definite. Then, however, stability of the null solution may be established on the semi-infinite time interval and without the boundedness assumption (3.1). These conclusions may be understood by observing that the cited authors establish the relations

$$F(t) \leq kE(t) = kE(0), \qquad 0 \leq t \leq \infty \tag{3.10}$$

for positive constant $k$. Thus, (3.1) is automatically satisfied and stability according to our definition follows from (3.9); in fact (3.10) shows that we may use the closed time interval in the definition. However, in this instance, convexity arguments are redundant since (3.10) already constitutes an adequate means of establishing stability on the semi-infinite time interval with respect to the measures $E(0)$ and $F(t)$.

Let us note that (3.9) forms the basis of a uniqueness theorem in linear elastodynamics due to Knops and Payne [4].

We now examine more carefully the relationship between the stability results of this paper and the more standard concepts of stability. For this purpose we introduce the appropriate form of a recent theorem of Gilbert and Knops [1] relating stability and

* Shield's [10] discussion of stability concerns measures different to those used here.

boundedness:

THEOREM 3.2. *Consider a class of solutions $u_i$ of* (2.1)–(2.3) *which belong to some linear vector space V. For elements of this class the following statements are equivalent:*

(i) *for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\int_{B(0)} \rho u_i u_i \, dv < \delta \Rightarrow \sup_{0 \le t < T} \int_{B(t)} \rho u_i u_i \, dv < \varepsilon,$$

*and*

(ii) *the condition*

$$\int_{B(0)} \rho u_i u_i \, dv < 1 \Rightarrow \sup_{0 \le t < T} \int_{B(t)} \rho u_i u_i \, dv < A$$

*where A is a finite positive constant.* The first case is a special case of our stability definition in which $Q = 0$.

Clearly then if $E(0) \le 0$ we may take $Q = 0$ in Theorem 3.1 and, provided we restrict our consideration to perturbations satisfying (3.1), it follows that Theorem 3.1 is equivalent to case (i). On the other hand one obtains from a series expansion of $\log F(t)$, using the fact that for $E(0) \le 0$,

$$\frac{d^2}{dt^2}[\log F(t)] \ge 0,$$

the inequality

$$F(t) \ge F(0)\, e^{[F'(0)/F(0)]t}. \tag{3.11}$$

Clearly without further restrictions on the $c_{ijkl}$, $\rho$, or the initial data (3.11) indicates that (ii) cannot be guaranteed. A specific example illustrating this point more clearly will be given presently. What we have shown is that if $E(0) \le Q$ (i) does not imply (ii) and we have an apparent contradiction to Theorem 3.2. There is, of course, an obvious answer to this contradiction, for if we require that all admissible perturbations satisfy (3.1) then the class of admissible solutions does not form a linear vector space. For clearly if $u_i$ belongs to the admissible class it does not follow that $\alpha u_i$ must belong to the class (for every real constant $\alpha$) nor that the sum of two admissible perturbations must be admissible.

We illustrate these remarks by giving a simple example which however is first used to show the significance of the half-open time interval in our stability definition. We suppose that there exists a positive constant $c_0$ such that

$$-c_{ijkl}\xi_{ij}\xi_{kl} \ge c_0 \xi_{ij}\xi_{ij}, \tag{3.12}$$

for all non-vanishing tensors $\xi_{ij}$. It is shown in the appendix that condition (3.12) is sufficient for instability of the null solution on the semi-infinite time interval whereas Theorem 3.1 states that if the perturbations are required to satisfy (3.1) then stability is insured in the half-open time interval. Now under mild conditions on the elasticities, it is known that with (3.12) satisfied there exist eigenvectors $w_i^n(\mathbf{x})$ with eigenvalues $\lambda_n$ satisfying

$$\begin{aligned} &\frac{\partial}{\partial x_j}\left\{c_{ijkl}\frac{\partial w_k^n}{\partial x_l}\right\} - \lambda_n^2 \rho w_i^n = 0 \qquad \text{in } B, \\ &w_i^n = 0 \qquad \text{on } \overline{\partial B_1}, \\ &c_{ijkl}\frac{\partial w_k^n}{\partial x_l} n_j = 0 \qquad \text{on } \partial B_2, \end{aligned} \tag{3.13}$$

(see, e.g. Gould [2]). It also follows from standard arguments that $\lambda_n \to \infty$ as $n \to \infty$. This fact clearly implies instability since for any $n$ the vector

$$u_i^n(\mathbf{x}, t) = \lambda_n^{-1} w_i^n e^{\lambda_n t}$$

is a solution of (2.1)–(2.3). By taking $n$ sufficiently large the initial data can be made arbitrarily small but the solution for subsequent $t$ will not depend continuously on the data in the sup, $L_2$, or energy norm.

Let us, however, consider the set of orthonormalized eigenvectors $w_i^n$, i.e.

$$\int_B \rho w_i^n w_i^m \, dv = \delta_{mn} \tag{3.14}$$

where $\delta_{mn}$ is the Kronecker delta. Clearly, the function

$$u_i^n(\mathbf{x}, t) = w_i^n e^{-\lambda_n(T-t)}$$

satisfies (2.1)–(2.3) and is bounded in the sense of (3.1). It follows from (3.13) that the total energy $E(0)$ vanishes and hence that $Q$ may be taken as zero. Thus

$$F(t) \equiv \int_{B(t)} \rho u_i^n u_i^n \, dv = e^{-2\lambda_n(T-t)}$$

or

$$\sup_{0 \le t \le T_1 < T} \int_{B(t)} \rho u_i^n u_i^n \, dv = e^{-2\lambda_n(T-T_1)}.$$

Since

$$F(0) = e^{-2\lambda_n T}$$

it follows that as $F(0) \to 0$, (i.e. $\lambda_n \to \infty$) then for $0 \le t \le T_1 < T$, $F(t)$ also tends to zero. In other words the null solution is stable in the half-open time interval. Note, however, that

$$F(T) = 1$$

and hence the solution is not stable in the closed interval $[0, T]$ if we use as our measure of stability

$$\sup_{0 \le t \le T} \int_{B(t)} \rho u_i^n u_i^n \, dv,$$

since $F(0) \to 0$ does not imply $F(t) \to 0$ for all $t \in [0, T]$.

We observe further that any function $u_i^n$ of the form $u_i^n(\mathbf{x}, t) = B_n w_i^n e^{\lambda_n t}$ with arbitrary real constant $B_n$ is also a solution of (2.1)–(2.3) for which

$$F(t) = F(0)e^{2\lambda_n t}. \tag{3.15}$$

Thus the condition $F(0) \leq 1$ certainly does not imply $F(t) < A$ in any fixed open time interval since by choosing $\lambda_n$ large enough the right hand side may be made arbitrarily large. This illustrates the remarks made following (3.11).

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## APPENDIX

To complete the analysis of this paper we derive by means of *a priori* inequalities conditions sufficient to ensure the instability of the null solution of (2.1)–(2.3) on the semi-infinite time interval. The following calculations show that for a negative-definite or indefinite strain energy certain classes of solutions must become unbounded with time. Instability of the null solution may then be deduced immediately.*†

Hence, let us first assume that

$$-\int_{B(t)} c_{ijkl}\xi_{ij}\xi_{kl}\,dv \geq c_0 \int_{B(t)} \xi_{ij}\xi_{ij}\,dv \tag{A1}$$

for some positive constant $c_0$ and non-vanishing tensor $\xi_{ij}$, and let us consider the function

$$F(t) = \int_{B(t)} \rho u_i u_i\,dv. \tag{A2}$$

* That a negative-definite strain energy is sufficient for instability of the null solution in a certain sense has been discussed by Kelvin [3], Koiter [6] and Movchan [8].

† A paper by Caughey Shield [11] has just come to the authors' attention in which inequalities are derived similar to those of this appendix.

Then, we have

$$\frac{d^2F}{dt^2} = 2\int_{B(t)} \left[\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} - c_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}\right] dv \geq 2c_0 \int_{B(t)} \rho\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} dv \tag{A3}$$

or

$$\frac{d^2F}{dt^2} \geq \frac{c_0\lambda_1}{\rho_{\max}} \int_{B(t)} \rho u_i u_i \, dv \tag{A4}$$

where (A4) is obtained from Poincaré's inequality, and $\lambda_1$ is the lowest eigenvalue of the corresponding membrane problem. In the case when the tractions are zero everywhere on the surface $\partial B$, we adjoin the normalization

$$\int_{B(t)} u_i \, dv = 0. \tag{A5}$$

On writing $\kappa^2 = c_0\lambda_1/\rho_{\max}$, we see that (A4) leads to

$$\int_{B(t)} \rho u_i u_i \, dv \geq \frac{2}{\kappa}\sinh \kappa t \int_{B(0)} \rho u_i \frac{\partial u_i}{\partial t} dv + \cosh \kappa t \int_{B(0)} \rho u_i u_i \, dv. \tag{A6}$$

Now let us examine those solutions whose initial displacement and velocity are such that the first integral on the right of (A6) is positive. Then, (A6) shows that these solutions have at least exponential time growth, from which it follows that the null solution is unstable; for if

$$\int_{B(0)} \rho u_i u_i \, dv + \int_{B(0)} \rho u_i \frac{\partial u_i}{\partial t} dv < \delta \tag{A7}$$

for any $\delta > 0$ it is always possible to choose $t$ so large that

$$\int_{B(t)} \rho u_i u_i \, dv > \varepsilon \tag{A8}$$

for any prescribed $\varepsilon$.

Next let us consider the case in which the strain energy expression is indefinite. However, let us restrict attention to the class of smooth initial data satisfying

$$\int_{B(0)} c_{ijkl}\frac{\partial^2 u_i}{\partial x_j \partial t}\frac{\partial^2 u_k}{\partial x_l \partial t} dv \leq 0 \tag{A9}$$

and

$$u_i(\mathbf{x}, 0) = 0. \tag{A10}$$

Then, for the function $F(t)$ defined in (A2) we have,

$$\frac{\mathrm{d}F}{\mathrm{d}t} = 2\int_0^t \int_{B(t)} \left[\rho \frac{\partial u_i}{\partial \eta}\frac{\partial u_i}{\partial \eta} - c_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}\right] \mathrm{d}v\,\mathrm{d}\eta$$

$$= 4\int_0^t \int_{B(t)} \rho \frac{\partial u_i}{\partial \eta}\frac{\partial u_i}{\partial \eta}\,\mathrm{d}v\,\mathrm{d}\eta - 2t\int_{B(0)} \rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}\,\mathrm{d}v \tag{A11}$$

where (A10) and the time independence of the total energy have been used. Now because

$$u_i(t) = \int_0^t \frac{\partial u_i}{\partial \eta}\,\mathrm{d}\eta \tag{A12}$$

we may use Schwarz's inequality to obtain

$$\int_{B(t)} \rho u_i u_i\,\mathrm{d}v \leq t\int_0^t \int_{B(t)} \rho \frac{\partial u_i}{\partial \eta}\frac{\partial u_i}{\partial \eta}\,\mathrm{d}v\,\mathrm{d}\eta \tag{A13}$$

and thus with (A11) derive

$$\frac{\mathrm{d}F}{\mathrm{d}t} \geq 4t^{-1}\mathrm{F} - 2t\int_{B(0)} \rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}\,\mathrm{d}v,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{F - t^2\int_{B(0)} \rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}\,\mathrm{d}v}{t^4}\right] \geq 0. \tag{A14}$$

An integration then gives

$$F \geq t^2 \int_{B(0)} \rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}\,\mathrm{d}v + t^4 \lim_{t\to 0}\left\{\frac{F - t^2\int_{B(0)} \rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}\,\mathrm{d}v}{t^4}\right\}. \tag{A15}$$

Under suitable smoothness assumptions on $u_i$, l'Hôpital's rule in conjunction with (A1) shows that the limit term in (A15) may be replaced by

$$-\frac{1}{3}\int_{B(0)} c_{ijkl}\frac{\partial^2 u_i}{\partial x_j \partial t}\frac{\partial^2 u_k}{\partial x_l \partial t}\,\mathrm{d}v, \tag{A16}$$

and hence on recalling (A9), we see that inequality (A15) yields

$$F \geq t^2 \int_{B(0)} \rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}\,\mathrm{d}v. \tag{A17}$$

Thus, those solutions with arbitrarily small initial kinetic energy and satisfying (A10) have a displacement whose mean square integral has at least a parabolic growth in time. This shows that the null solution is unstable in an obvious sense.

Finally, let us observe that (A17) is the best possible result within the stated class of displacements. This assertion is proved by taking the elasticities to vanish on the surface and putting

$$u_i = \varepsilon_i t \tag{A18}$$

for arbitrary constants $\varepsilon_i$. The displacement (A18) satisfies equation (2.1) and condition (A10), and is thus within the required class, while at the same time the boundary condition (2.3) is satisfied identically (we assume $\partial B \equiv \partial B_2$). On the other hand,

$$F(t) = \int_B \rho u_i u_i \,\mathrm{d}v = t^2 \int_B \rho \varepsilon_i \varepsilon_i \,\mathrm{d}v = t^2 \int_{B(0)} \rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}\,\mathrm{d}v.$$

Thus, for (A18) the inequality (A17) becomes an equality.

(*Received* 21 *March* 1968)

**Абстракт**—Даются условия, обеспечивающие непрерывную зависимость начального значения решения равновесия и некоторых других классов решения по отношению к упругодинамической началвной краеэй задаче. Метод испытания зависит от логарифмической выпуклости аргументов и заметный отсуствием никакого неопределенного условия упругостей. Для некоторых случаев, полученные результаты отличаются от соответствующих результатов, вытекающих из классической теории устойчивости Ляпунова. Даются примеры иллюстрироэающие эту разницу.